Logic seminar at NUS:
Computable models of small theories

Alex Gavruskin
(joint work with Bakh Khoussainov)

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

7th November 2012

Plan

1. The fundamental order
2. Decidable models of small theories
3. Computable models of small theories
4. Prospective applications: automatic structures

We consider only countable structures of countable languages.
And only *small* theories.

Definition

A first-order theory T is *small* if the set of finite first-order types of T without parameters, $S(T)$, is at most countable.

Let T be a small theory.

Fact

1. *T has a prime model and a saturated model.*
2. *If $p \in S(T)$ and $A \models p(\bar{a})$, then the theory $Th(A, \bar{a})$ has a prime model $(A_{\bar{a}}, \bar{c})$. Structures $A_{\bar{a}}$ are isomorphic for different A and $\bar{a}$. (Since* ***we consider structures up to isomorphism****, denote the structure by A_p.)*

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Call the structure $A_p \models T$ from Fact 2 *p-prime*, or *almost prime* if the type is not specified.

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Note

- The set $\mathcal{AP}_T$ of all almost prime models of a theory T is preordered under the relation $\preceq$ of elementary embeddability.
- $\mathcal{AP}_T / \sim$ is a poset, where $A \sim B \Leftrightarrow (A \preceq B \ \& \ B \preceq A)$.
- $(\mathcal{AP}_T / \sim, \preceq)$ has a unique least element—the prime model of T.

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Example

1. *A saturated structure is almost prime iff it is $\aleph_0$-categorical.*
2. *If a theory T is $\aleph_1$- but not $\aleph_0$-categorical then $\mathcal{AP}_T \cong \omega$.*
3. *If a theory T is Ehrenfeucht then $\mathcal{AP}_T$ has a max element.*

Proof.

If T is Ehrenfeucht then it has a non-principal *powerful*[*] type p. A p-prime structure is a maximal element of $\mathcal{AP}_T$. □

[*] A type p of a theory T is powerful if every model of T realising p realises every type of T as well.

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Proposition

If $A_p \sim A_q$ but $A_p \ncong A_q$, then there is a structure A such that $A \sim A_p$ but A is not almost prime.

Proof.

Form an elementary chain $A_0 \subseteq A_1 \subseteq \dots$ where $A_n \cong A_p$ if n is even and $A_n \cong A_q$ if n is odd. Put $A = \bigcup_{n \in \omega} A_n$. □

Note

The structure A can be presented as a union of an elementary chain of isomorphic almost prime structures, but A itself is not almost prime. Call such a structure *limit*.

Definition

A structure is *p-limit* (*limit*) if it is a union of an elementary chain of p-prime (isomorphic almost prime) structures but it itself is not p-prime (almost prime).

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Definition

A complete small theory T is an *AL theory* if every countable (unsaturated) model of T is either almost prime or limit.

Question

How far is the class of AL theories from the class of small theories?

Definition

A structure is *weakly limit* if it is the union of an elementary chain of almost prime structures.

Lemma (Sudoplatov 2004)

Every countable model of a small theory is either almost prime or weakly limit. □

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Note

- *A saturated structure is limit if and only if its theory has a non-principal powerful type, i. e. $\mathcal{AP}_T$ has a maximal element.*
- *Denote the set of all limit models of a theory T by $\mathcal{LS}_T$.*
- *The structure of the spectrum of an AL theory T is determined by a pre-order $\mathcal{AP}_T$ and a function $\lambda_T : \mathcal{AP}_T \to 2^{\mathcal{LS}_T}$ mapping a p-prime structure to the set of all p-limit structures.*
- *Think of λ_T as of a disjoint union of bipartite graphs. And draw a picture.*
- $\mathcal{LS}_T = \bigcup_{M \in \mathcal{AP}_T} \lambda_T(M).$

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Call the pair $(\mathcal{AP}_T, \lambda_T)$ *fundamental parameters* of T.

Example

1. $\aleph_0$-*categorical theories.* $\mathcal{AP}_T \cong 1$, $\lambda_T = \varnothing$.
2. $\aleph_1$-*categorical theories.* $\mathcal{AP}_T \cong \omega$, $\lambda_T = \varnothing$.
3. *Ehrenfeucht theories.*

   Ehrenfeucht:

$k = 3$ $\mathcal{AP}_T = \{0 < 1\}$, $\mathcal{LS}_T = \{a\}$, $\lambda_T(0) = \varnothing$, $\lambda_T(1) = \{a\}$;
$k \geq 3$ $\mathcal{AP}_T = \{0 < 1 \leq \ldots \leq k-2 \leq 1\}$, $\mathcal{LS}_T = \{a\}$,
$\lambda_T(0) = \varnothing$, $\lambda_T(1) = \ldots = \lambda_T(k-2) = \{a\}$.

   Morley–Lachlan:

$k = 6$ $\mathcal{AP}_T = \{0 < 1 < 2\}$, $\mathcal{LS}_T = \{a, b, c\}$,
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Proposition

If T is an AL theory then $\mathcal{AP}_T$ and λ_T satisfy the following:

1. $\mathcal{AP}_T$ *has a unique least element* A_0
2. $\lambda_T(A_0) = \varnothing$
3. *If* $\mathcal{AP}_T$ *has a maximal element* $Z_0 \neq A_0$ *then* $\lambda_T(Z_0) \neq \varnothing$
4. *If* $X_0 \not\sim X_1$ *are elements from* $\mathcal{AP}_T$ *then* $\lambda_T(X_0) \cap \lambda_T(X_1) = \varnothing$
5. *If* $X_0 \sim \ldots \sim X_{k+1}$ *is a maximal set of* $\sim$*-equivalent elements from* $\mathcal{AP}_T$ *then there is an element* M *such that* $M \in \bigcap_{0 \leq j \leq k+1} \lambda_T(X_j)$*, particularly,* $\lambda_T(X_j) \neq \varnothing$ □

Proposition

A theory is Ehrenfeucht if and only if 1–5 and:

6. *both* $\mathcal{AP}_T$ *and* λ_T *are finite*
7. $\mathcal{AP}_T$ *has a maximal element* $Z_0 \neq A_0$ □

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Problem

Let $\mathcal{AP}$ be a partial order, $\mathcal{LS}$ be a set, and $\lambda : \mathcal{AP} \to 2^{\mathcal{LS}}$. What properties, in addition to 1–5, must be satisfied in order to guarantee the existence of an AL theory T such that $\varphi : \mathcal{AP} \cong \mathcal{AP}_T$, $\psi : \mathcal{LS} \cong \mathcal{LS}_T$, and $\lambda_T\varphi(X) = \{\psi(M) \mid M \in \lambda(X)\}$ for every $X \in \mathcal{AP}$.

Problem

The same for properties 1–7 and Ehrenfeucht theories.

Approaches:

1. Sudoplatov, Complete theories with **finitely many** countable models II, *Algebra and Logic*, 45, 3, 180–200, 2006.
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If $(\mathcal{AP}_T, \preceq)$ contains a sub-order of the type of $\omega + 1$, T can not be an AL theory.

Proof.

Take the structures corresponding to the ω, say, $A_0 \subseteq A_1 \subseteq \dots$, take a union of the chain, say, A. It is neither almost prime nor limit. Since A is not universal, it can not be saturated. □

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Theorem (G, Khoussainov 2012)

Let $\mathcal{L}$ be a finite lattice. Then there exists an AL theory T such that the fundamental order of T is $\mathcal{L}$, that is, $(\mathcal{AP}_T/\sim, \preceq) \cong \mathcal{L}$.

This is the first part of the theorem. See Part 2 of the talk for the second part of the theorem and for the idea of proof.

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Part 2

Decidable models of AL theories

Fact

Henkin construction provides us with a decidable model of a decidable consistent theory.

How does this model look like?

Open problem (1973)

Is the prime model of a decidable strongly small theory decidable?

Theorem (Millar 1983—year of my birth)

Every decidable small theory has a decidable almost prime model.

1. Every countable model of a decidable $\aleph_0$-categorical theory is decidable.
2. Harrington, Khissamiev: Every countable model of a decidable $\aleph_1$-categorical theory is decidable.
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The class of small theories is a really bad one.

Theorem (Millar 1978)

There exists a decidable small theory T whose types are all decidable but T does not have a decidable saturated model.

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There exists a decidable small theory T in finite language whose types are all decidable but T does not have a decidable saturated model.

Corollary

There exists a prime structure of finite language such that it has an X-computable presentation if and only if X is not computable.

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Theorem (G 2011)

If an AL theory T is decidable then T has a decidable prime model.

Idea of proof.

Omit decidably as many types as possible. The main tool for that is the next theorem. □

Theorem (Millar 1983)

Let T be a decidable theory, S a Σ^0_2-set of decidable (complete) types of T. Then, uniformly in T and a Σ^0_2-index for S, there is a decidable model of T omitting all the non-principal types in S.

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Goncharov-Millar 1973-1986 Theorem does not hold in the world of AL theories:

Corollary 1

If T is a decidable AL theory all whose types are decidable, then every homogeneous model of T is decidable. Particularly, the saturated model is decidable.

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Definition

Let T be an AL theory with fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. One can naturally define the sub-parameters $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$ corresponding to decidable models of T. Call these sub-parameters *spectra of decidable models* of the theory T.

Question (Spectral problem in the class of decidable presentations)

Let $\mathbb{K}$ be a class of theories with fixed fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of decidable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of decidable models $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$?

Example

If $\mathbb{K}$ is a class of $\aleph$-categorical theories then the spectral problem is trivial due to Harrington and Khissamiev. Spectra of decidable models of a theory T from $\mathbb{K}$ is either empty or coincide with T's spectral parameters.

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Corollary 3

If T is an AL theory then $\mathcal{AP}_T^{\mathcal{D}}$ is downward closed in $\mathcal{AP}_T$.

Corollary 4

If A is a decidable p-limit structure then every p-prime structure is decidable.

Problem

Let A and B be p-limit structures such that B is decidable. Is A decidable?

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Let A and B be p-limit structures such that B is decidable. Is A decidable?

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Let $\mathcal{L}$ be a finite lattice and $\mathcal{L}'$ be its sublattice. Suppose $\mathcal{L}'$ is downward closed in $\mathcal{L}$. Then there exists an AL theory T such that:

1. *The fundamental order of T is $\mathcal{L}$, that is, $(\mathcal{AP}_T / \sim, \preceq) \cong \mathcal{L}$*
2. *The spectra of decibel models of T is $\mathcal{L}'$, that is, $\mathcal{AP}_T^{\mathcal{D}} \cong \mathcal{L}'$.*

Idea of proof.

Take a theory T_0 having a non-principal type p. Take the lattice $\mathcal{L}$. For elements $a <_{\mathcal{L}} b$, say that if b "makes" p realised, so does a. For elements $c = lub_{\mathcal{L}}\{a, b\}$, say that if both a and b "make" p realised, then so does c. Satisfy these conditions in the freest possible way. Use amalgamation construction of course.

Since $\mathcal{L}'$ is downward closed, there is an element a such that $\mathcal{L}' = \{x \mid x \leq_{\mathcal{L}} a\}$. Make the type corresponding to a decidable and leave undecidable as many types as possible. □

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Part 3

Spectra of computable models of AL theories

Question (Spectral problem in the class of computable presentations)

Let $\mathcal{K}$ be a class of theories with fixed spectral parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of computable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of computable models?

1. $\aleph_1$-categorical theories. The problem is quite complicated. An upper bound for the complexity of spectra is $\Sigma^0_3(\varnothing^\omega)$ (Nies). All known spectra are finite of co-finite (Many people from the US and Russia).
2. Ehrenfeucht theories. There are very many examples of the spectra. For instance, the spectra are not necessarily downward closed:

Theorem (G 2010)

There exists an Ehrenfeucht theory T and structures $A, B \in \mathcal{AP}_T$ such that $B \preceq A$, A is computable, B is not computable. Moreover, these structures can be chosen to be $\sim$-equivalent.

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The situation in the class of computable presentations is completely different from the one in the class of decidable presentation:

Theorem (Khoussainov, Nies, Shore 1997)

There exists an Ehrenfeucht theory T having exactly 3 countable models such that the only computable model is saturated.

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Part 4

Spectra of automatic models of AL theories

Definition

A structure A of a finite predicate language is *automatic* if its domain and relations are recognisable by (finite string) automata.

Theorem (Hodgson 1976)

Automatic models are decidable.

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Let $\mathcal{K}$ be a class of theories with fixed spectral parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of automatic models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of automatic models?

Theorem (Semukhin, Stephan 2010)

1. *There is an $\aleph_1$-categorical theory the only automatic model of which is prime.*
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Theorem (G, Gavruskina 2011)

If an Ehrenfeucht theory T has infinite $dcl(\varnothing)$ and exactly 3 models, one of which is automatic, then all the models of T are automatic.

Proof.

Follows from the following two theorems. □

Theorem (Gavruskina 2010)

Let T be a variant of Ehrenfeucht's or Peretyatkin's example having an automatic model. Then all the models of T are automatic.

Theorem (Tanović 2007)

If an Ehrenfeucht theory T has infinite $dcl(\varnothing)$ and exactly 3 models, then T interprets a variant of Ehrenfeucht's or Peretyatkin's example.

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Thank you for your attention!

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