

Logic seminar at NUS: *Computable models of small theories*

Alex Gavruskin
(joint work with Bakh Khossainov)



THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

7th November 2012

- ① The fundamental order
- ② Decidable models of small theories
- ③ Computable models of small theories
- ④ Prospective applications: automatic structures

We consider only countable structures of countable languages.
And only *small* theories.

Definition

A first-order theory T is *small* if the set of finite first-order types of T without parameters, $S(T)$, is at most countable.

Let T be a small theory.

Fact

- 1 T has a prime model and a saturated model.
- 2 If $p \in S(T)$ and $A \models p(\bar{a})$, then the theory $\text{Th}(A, \bar{a})$ has a prime model $(A_{\bar{a}}, \bar{c})$. Structures $A_{\bar{a}}$ are isomorphic for different A and \bar{a} . (Since **we consider structures up to isomorphism**, denote the structure by A_p .)

Definition

Call the structure $A_p \models T$ from Fact 2 *p*-prime, or *almost prime* if the type is not specified.

We consider only countable structures of countable languages.
And only *small* theories.

Definition

A first-order theory T is *small* if the set of finite first-order types of T without parameters, $S(T)$, is at most countable.

Let T be a small theory.

Fact

- 1 T has a prime model and a saturated model.
- 2 If $p \in S(T)$ and $A \models p(\bar{a})$, then the theory $\text{Th}(A, \bar{a})$ has a prime model $(A_{\bar{a}}, \bar{c})$. Structures $A_{\bar{a}}$ are isomorphic for different A and \bar{a} . (Since **we consider structures up to isomorphism**, denote the structure by A_p .)

Definition

Call the structure $A_p \models T$ from Fact 2 *p*-prime, or *almost prime* if the type is not specified.

We consider only countable structures of countable languages.
And only *small* theories.

Definition

A first-order theory T is *small* if the set of finite first-order types of T without parameters, $S(T)$, is at most countable.

Let T be a small theory.

Fact

- 1 T has a prime model and a saturated model.
- 2 If $p \in S(T)$ and $A \models p(\bar{a})$, then the theory $\text{Th}(A, \bar{a})$ has a prime model $(A_{\bar{a}}, \bar{c})$. Structures $A_{\bar{a}}$ are isomorphic for different A and \bar{a} . (Since **we consider structures up to isomorphism**, denote the structure by A_p .)

Definition

Call the structure $A_p \models T$ from Fact 2 *p*-prime, or *almost prime* if the type is not specified.

Note

- The set \mathcal{AP}_T of all almost prime models of a theory T is preordered under the relation \preceq of elementary embeddability.
- \mathcal{AP}_T / \sim is a poset, where $A \sim B \Leftrightarrow (A \preceq B \ \& \ B \preceq A)$.
- $(\mathcal{AP}_T / \sim, \preceq)$ has a unique least element—the prime model of T .

Definition

We call the partial order $(\mathcal{AP}_T / \sim, \preceq)$ the *fundamental order* of the theory T .

Note

- The set \mathcal{AP}_T of all almost prime models of a theory T is preordered under the relation \preceq of elementary embeddability.
- \mathcal{AP}_T / \sim is a poset, where $A \sim B \Leftrightarrow (A \preceq B \ \& \ B \preceq A)$.
- $(\mathcal{AP}_T / \sim, \preceq)$ has a unique least element—the prime model of T .

Definition

We call the partial order $(\mathcal{AP}_T / \sim, \preceq)$ the *fundamental order* of the theory T .

Example

- 1 A saturated structure is almost prime iff it is \aleph_0 -categorical.
- 2 If a theory T is \aleph_1 - but not \aleph_0 -categorical then $\mathcal{AP}_T \cong \omega$.
- 3 If a theory T is Ehrenfeucht then \mathcal{AP}_T has a max element.

Proof.

If T is Ehrenfeucht then it has a non-principal *powerful*^{*} type p .
A p -prime structure is a maximal element of \mathcal{AP}_T . \square

^{*}A type p of a theory T is powerful if every model of T realising p realises every type of T as well.

Example

- 1 A saturated structure is almost prime iff it is \aleph_0 -categorical.
- 2 If a theory T is \aleph_1 - but not \aleph_0 -categorical then $\mathcal{AP}_T \cong \omega$.
- 3 If a theory T is Ehrenfeucht then \mathcal{AP}_T has a max element.

Proof.

If T is Ehrenfeucht then it has a non-principal *powerful*^{*} type p .
A p -prime structure is a maximal element of \mathcal{AP}_T . □

^{*}A type p of a theory T is powerful if every model of T realising p realises every type of T as well.

Proposition

If $A_p \sim A_q$ but $A_p \not\cong A_q$, then there is a structure A such that $A \sim A_p$ but A is not almost prime.

Proof.

Form an elementary chain $A_0 \subseteq A_1 \subseteq \dots$ where $A_n \cong A_p$ if n is even and $A_n \cong A_q$ if n is odd. Put $A = \bigcup_{n \in \omega} A_n$. \square

Note

The structure A can be presented as a union of an elementary chain of isomorphic almost prime structures, but A itself is not almost prime. Call such a structure *limit*.

Definition

A structure is p -*limit* (*limit*) if it is a union of an elementary chain of p -prime (isomorphic almost prime) structures but it itself is not p -prime (almost prime).

Proposition

If $A_p \sim A_q$ but $A_p \not\cong A_q$, then there is a structure A such that $A \sim A_p$ but A is not almost prime.

Proof.

Form an elementary chain $A_0 \subseteq A_1 \subseteq \dots$ where $A_n \cong A_p$ if n is even and $A_n \cong A_q$ if n is odd. Put $A = \bigcup_{n \in \omega} A_n$. □

Note

The structure A can be presented as a union of an elementary chain of isomorphic almost prime structures, but A itself is not almost prime. Call such a structure *limit*.

Definition

A structure is p -*limit* (*limit*) if it is a union of an elementary chain of p -prime (isomorphic almost prime) structures but it itself is not p -prime (almost prime).

Proposition

If $A_p \sim A_q$ but $A_p \not\cong A_q$, then there is a structure A such that $A \sim A_p$ but A is not almost prime.

Proof.

Form an elementary chain $A_0 \subseteq A_1 \subseteq \dots$ where $A_n \cong A_p$ if n is even and $A_n \cong A_q$ if n is odd. Put $A = \bigcup_{n \in \omega} A_n$. □

Note

The structure A can be presented as a union of an elementary chain of isomorphic almost prime structures, but A itself is not almost prime. Call such a structure *limit*.

Definition

A structure is p -*limit* (*limit*) if it is a union of an elementary chain of p -prime (isomorphic almost prime) structures but it itself is not p -prime (almost prime).

Definition

A complete small theory T is an *AL theory* if every countable (unsaturated) model of T is either almost prime or limit.

Question

How far is the class of AL theories from the class of small theories?

Definition

A structure is *weakly limit* if it is the union of an elementary chain of almost prime structures.

Lemma (Sudoplatov 2004)

Every countable model of a small theory is either almost prime or weakly limit. □

Definition

A complete small theory T is an *AL theory* if every countable (unsaturated) model of T is either almost prime or limit.

Question

How far is the class of AL theories from the class of small theories?

Definition

A structure is *weakly limit* if it is the union of an elementary chain of almost prime structures.

Lemma (Sudoplatov 2004)

Every countable model of a small theory is either almost prime or weakly limit. □

Definition

A complete small theory T is an *AL theory* if every countable (unsaturated) model of T is either almost prime or limit.

Question

How far is the class of AL theories from the class of small theories?

Definition

A structure is *weakly limit* if it is the union of an elementary chain of almost prime structures.

Lemma (Sudoplatov 2004)

Every countable model of a small theory is either almost prime or weakly limit. □

Note

- A saturated structure is limit if and only if its theory has a non-principal powerful type, i. e. \mathcal{AP}_T has a maximal element.
- Denote the set of all limit models of a theory T by \mathcal{LS}_T .
- The structure of the spectrum of an AL theory T is determined by a pre-order \mathcal{AP}_T and a function $\lambda_T : \mathcal{AP}_T \rightarrow 2^{\mathcal{LS}_T}$ mapping a p -prime structure to the set of all p -limit structures.
- Think of λ_T as of a disjoint union of bipartite graphs. And draw a picture.
- $\mathcal{LS}_T = \bigcup_{M \in \mathcal{AP}_T} \lambda_T(M)$.

Note

- A saturated structure is limit if and only if its theory has a non-principal powerful type, i. e. \mathcal{AP}_T has a maximal element.
- Denote the set of all limit models of a theory T by \mathcal{LS}_T .
- The structure of the spectrum of an AL theory T is determined by a pre-order \mathcal{AP}_T and a function $\lambda_T : \mathcal{AP}_T \rightarrow 2^{\mathcal{LS}_T}$ mapping a p -prime structure to the set of all p -limit structures.
- Think of λ_T as of a disjoint union of bipartite graphs. And draw a picture.
- $\mathcal{LS}_T = \bigcup_{M \in \mathcal{AP}_T} \lambda_T(M)$.

Definition

Call the pair $(\mathcal{AP}_T, \lambda_T)$ *fundamental parameters* of T .

Example

- 1 \aleph_0 -categorical theories. $\mathcal{AP}_T \cong 1$, $\lambda_T = \emptyset$.
- 2 \aleph_1 -categorical theories. $\mathcal{AP}_T \cong \omega$, $\lambda_T = \emptyset$.
- 3 Ehrenfeucht theories.

Ehrenfeucht:

$k = 3$ $\mathcal{AP}_T = \{0 < 1\}$, $\mathcal{LS}_T = \{a\}$, $\lambda_T(0) = \emptyset$, $\lambda_T(1) = \{a\}$;

$k \geq 3$ $\mathcal{AP}_T = \{0 < 1 \leq \dots \leq k-2 \leq 1\}$, $\mathcal{LS}_T = \{a\}$,
 $\lambda_T(0) = \emptyset$, $\lambda_T(1) = \dots = \lambda_T(k-2) = \{a\}$.

Morley–Lachlan:

$k = 6$ $\mathcal{AP}_T = \{0 < 1 < 2\}$, $\mathcal{LS}_T = \{a, b, c\}$,
 $\lambda_T(0) = \lambda_T(1) = \emptyset$, $\lambda_T(2) = \{a, b, c\}$.

Definition

Call the pair $(\mathcal{AP}_T, \lambda_T)$ *fundamental parameters* of T .

Example

- 1 \aleph_0 -categorical theories. $\mathcal{AP}_T \cong 1$, $\lambda_T = \emptyset$.
- 2 \aleph_1 -categorical theories. $\mathcal{AP}_T \cong \omega$, $\lambda_T = \emptyset$.
- 3 Ehrenfeucht theories.

Ehrenfeucht:

$k = 3$ $\mathcal{AP}_T = \{0 < 1\}$, $\mathcal{LS}_T = \{a\}$, $\lambda_T(0) = \emptyset$, $\lambda_T(1) = \{a\}$;

$k \geq 3$ $\mathcal{AP}_T = \{0 < 1 \leq \dots \leq k-2 \leq 1\}$, $\mathcal{LS}_T = \{a\}$,
 $\lambda_T(0) = \emptyset$, $\lambda_T(1) = \dots = \lambda_T(k-2) = \{a\}$.

Morley–Lachlan:

$k = 6$ $\mathcal{AP}_T = \{0 < 1 < 2\}$, $\mathcal{LS}_T = \{a, b, c\}$,
 $\lambda_T(0) = \lambda_T(1) = \emptyset$, $\lambda_T(2) = \{a, b, c\}$.

Proposition

If T is an AL theory then \mathcal{AP}_T and λ_T satisfy the following:

- 1 \mathcal{AP}_T has a unique least element A_0
- 2 $\lambda_T(A_0) = \emptyset$
- 3 If \mathcal{AP}_T has a maximal element $Z_0 \neq A_0$ then $\lambda_T(Z_0) \neq \emptyset$
- 4 If $X_0 \not\sim X_1$ are elements from \mathcal{AP}_T then $\lambda_T(X_0) \cap \lambda_T(X_1) = \emptyset$
- 5 If $X_0 \sim \dots \sim X_{k+1}$ is a maximal set of \sim -equivalent elements from \mathcal{AP}_T then there is an element M such that $M \in \bigcap_{0 \leq j \leq k+1} \lambda_T(X_j)$, particularly, $\lambda_T(X_j) \neq \emptyset$



Proposition

A theory is Ehrenfeucht if and only if 1–5 and:

- 6 both \mathcal{AP}_T and λ_T are finite
- 7 \mathcal{AP}_T has a maximal element $Z_0 \neq A_0$



Proposition

If T is an AL theory then \mathcal{AP}_T and λ_T satisfy the following:

- 1 \mathcal{AP}_T has a unique least element A_0
- 2 $\lambda_T(A_0) = \emptyset$
- 3 If \mathcal{AP}_T has a maximal element $Z_0 \neq A_0$ then $\lambda_T(Z_0) \neq \emptyset$
- 4 If $X_0 \not\sim X_1$ are elements from \mathcal{AP}_T then $\lambda_T(X_0) \cap \lambda_T(X_1) = \emptyset$
- 5 If $X_0 \sim \dots \sim X_{k+1}$ is a maximal set of \sim -equivalent elements from \mathcal{AP}_T then there is an element M such that $M \in \bigcap_{0 \leq j \leq k+1} \lambda_T(X_j)$, particularly, $\lambda_T(X_j) \neq \emptyset$



Proposition

A theory is Ehrenfeucht if and only if 1–5 and:

- 6 both \mathcal{AP}_T and λ_T are finite
- 7 \mathcal{AP}_T has a maximal element $Z_0 \neq A_0$



Problem

Let \mathcal{AP} be a partial order, \mathcal{LS} be a set, and $\lambda : \mathcal{AP} \rightarrow 2^{\mathcal{LS}}$. What properties, in addition to 1–5, must be satisfied in order to guarantee the existence of an AL theory T such that $\varphi : \mathcal{AP} \cong \mathcal{AP}_T$, $\psi : \mathcal{LS} \cong \mathcal{LS}_T$, and $\lambda_T \varphi(X) = \{\psi(M) \mid M \in \lambda(X)\}$ for every $X \in \mathcal{AP}$.

Problem

The same for properties 1–7 and Ehrenfeucht theories.

Approaches:

- 1 Sudoplatov, Complete theories with **finitely many** countable models II, *Algebra and Logic*, 45, 3, 180–200, 2006.
- 2 Recent papers at <http://sites.google.com/site/gavruskin/publications/>

Problem

Let \mathcal{AP} be a partial order, \mathcal{LS} be a set, and $\lambda : \mathcal{AP} \rightarrow 2^{\mathcal{LS}}$. What properties, in addition to 1–5, must be satisfied in order to guarantee the existence of an AL theory T such that $\varphi : \mathcal{AP} \cong \mathcal{AP}_T$, $\psi : \mathcal{LS} \cong \mathcal{LS}_T$, and $\lambda_T \varphi(X) = \{\psi(M) \mid M \in \lambda(X)\}$ for every $X \in \mathcal{AP}$.

Problem

The same for properties 1–7 and Ehrenfeucht theories.

Approaches:

- 1 Sudoplatov, Complete theories with **finitely many** countable models II, *Algebra and Logic*, 45, 3, 180–200, 2006.
- 2 Recent papers at <http://sites.google.com/site/gavruskin/publications/>

Problem

Let \mathcal{AP} be a partial order, \mathcal{LS} be a set, and $\lambda : \mathcal{AP} \rightarrow 2^{\mathcal{LS}}$. What properties, in addition to 1–5, must be satisfied in order to guarantee the existence of an AL theory T such that $\varphi : \mathcal{AP} \cong \mathcal{AP}_T$, $\psi : \mathcal{LS} \cong \mathcal{LS}_T$, and $\lambda_T \varphi(X) = \{\psi(M) \mid M \in \lambda(X)\}$ for every $X \in \mathcal{AP}$.

Problem

The same for properties 1–7 and Ehrenfeucht theories.

Approaches:

- 1 Sudoplatov, Complete theories with **finitely many** countable models II, *Algebra and Logic*, 45, 3, 180–200, 2006.
- 2 Recent papers at <http://sites.google.com/site/gavruskin/publications/>

Problem

Let \mathcal{AP} be a partial order, \mathcal{LS} be a set, and $\lambda : \mathcal{AP} \rightarrow 2^{\mathcal{LS}}$. What properties, in addition to 1–5, must be satisfied in order to guarantee the existence of an AL theory T such that $\varphi : \mathcal{AP} \cong \mathcal{AP}_T$, $\psi : \mathcal{LS} \cong \mathcal{LS}_T$, and $\lambda_T \varphi(X) = \{\psi(M) \mid M \in \lambda(X)\}$ for any $X \in \mathcal{AP}$.

Proposition

If $(\mathcal{AP}_T, \preceq)$ contains a sub-order of the type of $\omega + 1$, T can not be an AL theory.

Proof.

Take the structures corresponding to the ω , say, $A_0 \subseteq A_1 \subseteq \dots$, take a union of the chain, say, A . It is neither almost prime nor limit. Since A is not universal, it can not be saturated. \square

Problem

Let \mathcal{AP} be a partial order, \mathcal{LS} be a set, and $\lambda : \mathcal{AP} \rightarrow 2^{\mathcal{LS}}$. What properties, in addition to 1–5, must be satisfied in order to guarantee the existence of an AL theory T such that $\varphi : \mathcal{AP} \cong \mathcal{AP}_T$, $\psi : \mathcal{LS} \cong \mathcal{LS}_T$, and $\lambda_T \varphi(X) = \{\psi(M) \mid M \in \lambda(X)\}$ for any $X \in \mathcal{AP}$.

Proposition

If $(\mathcal{AP}_T, \preceq)$ contains a sub-order of the type of $\omega + 1$, T can not be an AL theory.

Proof.

Take the structures corresponding to the ω , say, $A_0 \subseteq A_1 \subseteq \dots$, take a union of the chain, say, A . It is neither almost prime nor limit. Since A is not universal, it can not be saturated. \square

Theorem (G, Khoussainov 2012)

Let \mathcal{L} be a finite lattice. Then there exists an AL theory T such that the fundamental order of T is \mathcal{L} , that is, $(\mathcal{AP}_T / \sim, \preceq) \cong \mathcal{L}$.

This is the first part of the theorem. See Part 2 of the talk for the second part of the theorem and for the idea of proof.

Theorem (G, Khoussainov 2012)

Let \mathcal{L} be a finite lattice. Then there exists an AL theory T such that the fundamental order of T is \mathcal{L} , that is, $(\mathcal{AP}_T / \sim, \preceq) \cong \mathcal{L}$.

This is the first part of the theorem. See Part 2 of the talk for the second part of the theorem and for the idea of proof.

Part 2

Decidable models of AL theories

Fact

Henkin construction provides us with a decidable model of a decidable consistent theory.

How does this model look like?

Open problem (1973)

Is the prime model of a decidable strongly small theory decidable?

Theorem (Millar 1983—year of my birth)

Every decidable small theory has a decidable almost prime model.

- 1 Every countable model of a decidable \aleph_0 -categorical theory is decidable.
- 2 Harrington, Khissamiev: Every countable model of a decidable \aleph_1 -categorical theory is decidable.
- 3 Prime models of decidable Ehrenfeucht theories are decidable.
Morley, Lachlan, and Peretyatkin: This is the best possible result.

Fact

Henkin construction provides us with a decidable model of a decidable consistent theory.

How does this model look like?

Open problem (1973)

Is the prime model of a decidable strongly small theory decidable?

Theorem (Millar 1983—year of my birth)

Every decidable small theory has a decidable almost prime model.

- 1 Every countable model of a decidable \aleph_0 -categorical theory is decidable.
- 2 Harrington, Khissamiev: Every countable model of a decidable \aleph_1 -categorical theory is decidable.
- 3 Prime models of decidable Ehrenfeucht theories are decidable.
Morley, Lachlan, and Peretyatkin: This is the best possible result.

Fact

Henkin construction provides us with a decidable model of a decidable consistent theory.

How does this model look like?

Open problem (1973)

Is the prime model of a decidable strongly small theory decidable?

Theorem (Millar 1983—year of my birth)

Every decidable small theory has a decidable almost prime model.

- 1 Every countable model of a decidable \aleph_0 -categorical theory is decidable.
- 2 Harrington, Khissamiev: Every countable model of a decidable \aleph_1 -categorical theory is decidable.
- 3 Prime models of decidable Ehrenfeucht theories are decidable.
Morley, Lachlan, and Peretyatkin: This is the best possible result.

Fact

Henkin construction provides us with a decidable model of a decidable consistent theory.

How does this model look like?

Open problem (1973)

Is the prime model of a decidable strongly small theory decidable?

Theorem (Millar 1983—year of my birth)

Every decidable small theory has a decidable almost prime model.

- 1 Every countable model of a decidable \aleph_0 -categorical theory is decidable.
- 2 Harrington, Khissamiev: Every countable model of a decidable \aleph_1 -categorical theory is decidable.
- 3 Prime models of decidable Ehrenfeucht theories are decidable.
Morley, Lachlan, and Peretyatkin: This is the best possible result.

Theorem (Goncharov, Millar 1973-1986)

The class of small theories is a really bad one.

Theorem (Millar 1978)

There exists a decidable small theory T whose types are all decidable but T does not have a decidable saturated model.

Theorem (G, Khoussainov 2012)

There exists a decidable small theory T in finite language whose types are all decidable but T does not have a decidable saturated model.

Corollary

There exists a prime structure of finite language such that it has an X -computable presentation if and only if X is not computable.

Theorem (Goncharov, Millar 1973-1986)

The class of small theories is a really bad one.

Theorem (Millar 1978)

There exists a decidable small theory T whose types are all decidable but T does not have a decidable saturated model.

Theorem (G, Khoussainov 2012)

There exists a decidable small theory T in finite language whose types are all decidable but T does not have a decidable saturated model.

Corollary

There exists a prime structure of finite language such that it has an X -computable presentation if and only if X is not computable.

Theorem (Goncharov, Millar 1973-1986)

The class of small theories is a really bad one.

Theorem (Millar 1978)

There exists a decidable small theory T whose types are all decidable but T does not have a decidable saturated model.

Theorem (G, Khoussainov 2012)

There exists a decidable small theory T in finite language whose types are all decidable but T does not have a decidable saturated model.

Corollary

There exists a prime structure of finite language such that it has an X -computable presentation if and only if X is not computable.

Theorem (Goncharov, Millar 1973-1986)

The class of small theories is a really bad one.

Theorem (Millar 1978)

There exists a decidable small theory T whose types are all decidable but T does not have a decidable saturated model.

Theorem (G, Khousseinov 2012)

There exists a decidable small theory T in finite language whose types are all decidable but T does not have a decidable saturated model.

Corollary

There exists a prime structure of finite language such that it has an X -computable presentation if and only if X is not computable.

Theorem (G 2011)

If an AL theory T is decidable then T has a decidable prime model.

Idea of proof.

Omit decidable as many types as possible. The main tool for that is the next theorem. □

Theorem (Millar 1983)

Let T be a decidable theory, S a Σ_2^0 -set of decidable (complete) types of T . Then, uniformly in T and a Σ_2^0 -index for S , there is a decidable model of T omitting all the non-principal types in S .

Theorem (G 2011)

If an AL theory T is decidable then T has a decidable prime model.

Idea of proof.

Omit decidable as many types as possible. The main tool for that is the next theorem. □

Theorem (Millar 1983)

Let T be a decidable theory, S a Σ_2^0 -set of decidable (complete) types of T . Then, uniformly in T and a Σ_2^0 -index for S , there is a decidable model of T omitting all the non-principal types in S .

Goncharov-Millar 1973-1986 Theorem does not hold in the world of AL theories:

Corollary 1

If T is a decidable AL theory all whose types are decidable, then every homogeneous model of T is decidable. Particularly, the saturated model is decidable.

Corollary 2

If an AL theory T has a decidable saturated model then all homogeneous models of T are decidable.

Goncharov-Millar 1973-1986 Theorem does not hold in the world of AL theories:

Corollary 1

If T is a decidable AL theory all whose types are decidable, then every homogeneous model of T is decidable. Particularly, the saturated model is decidable.

Corollary 2

If an AL theory T has a decidable saturated model then all homogeneous models of T are decidable.

Goncharov-Millar 1973-1986 Theorem does not hold in the world of AL theories:

Corollary 1

If T is a decidable AL theory all whose types are decidable, then every homogeneous model of T is decidable. Particularly, the saturated model is decidable.

Corollary 2

If an AL theory T has a decidable saturated model then all homogeneous models of T are decidable.

Definition

Let T be an AL theory with fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. One can naturally define the sub-parameters $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$ corresponding to decidable models of T . Call these sub-parameters *spectra of decidable models* of the theory T .

Question (Spectral problem in the class of decidable presentations)

Let \mathcal{K} be a class of theories with fixed fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of decidable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of decidable models $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$?

Example

If \mathcal{K} is a class of \aleph -categorical theories then the spectral problem is trivial due to Harrington and Khissamiev. Spectra of decidable models of a theory T from \mathcal{K} is either empty or coincide with T 's spectral parameters.

Definition

Let T be an AL theory with fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. One can naturally define the sub-parameters $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$ corresponding to decidable models of T . Call these sub-parameters *spectra of decidable models* of the theory T .

Question (Spectral problem in the class of decidable presentations)

Let \mathcal{K} be a class of theories with fixed fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of decidable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of decidable models $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$?

Example

If \mathcal{K} is a class of \aleph -categorical theories then the spectral problem is trivial due to Harrington and Khissamiev. Spectra of decidable models of a theory T from \mathcal{K} is either empty or coincide with T 's spectral parameters.

Definition

Let T be an AL theory with fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. One can naturally define the sub-parameters $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$ corresponding to decidable models of T . Call these sub-parameters *spectra of decidable models* of the theory T .

Question (Spectral problem in the class of decidable presentations)

Let \mathcal{K} be a class of theories with fixed fundamental parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of decidable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of decidable models $(\mathcal{AP}_T^{\mathcal{D}}, \lambda_T^{\mathcal{D}})$?

Example

If \mathcal{K} is a class of \aleph -categorical theories then the spectral problem is trivial due to Harrington and Khissamiev. Spectra of decidable models of a theory T from \mathcal{K} is either empty or coincide with T 's spectral parameters.

Corollary 3

If T is an AL theory then $\mathcal{AP}_T^{\mathcal{D}}$ is downward closed in \mathcal{AP}_T .

Corollary 4

If A is a decidable p -limit structure then every p -prime structure is decidable.

Problem

Let A and B be p -limit structures such that B is decidable. Is A decidable?

Corollary 3

If T is an AL theory then $\mathcal{AP}_T^{\mathcal{D}}$ is downward closed in \mathcal{AP}_T .

Corollary 4

If A is a decidable p -limit structure then every p -prime structure is decidable.

Problem

Let A and B be p -limit structures such that B is decidable. Is A decidable?

Corollary 3

If T is an AL theory then \mathcal{AP}_T^D is downward closed in \mathcal{AP}_T .

Corollary 4

If A is a decidable p -limit structure then every p -prime structure is decidable.

Problem

Let A and B be p -limit structures such that B is decidable. Is A decidable?

Theorem (G, Khoussainov 2012)

Let \mathcal{L} be a finite lattice and \mathcal{L}' be its sublattice. Suppose \mathcal{L}' is downward closed in \mathcal{L} . Then there exists an AL theory T such that:

- 1 The fundamental order of T is \mathcal{L} , that is, $(\mathcal{AP}_T / \sim, \preceq) \cong \mathcal{L}$
- 2 The spectra of decibel models of T is \mathcal{L}' , that is, $\mathcal{AP}_T^D \cong \mathcal{L}'$.

Idea of proof.

Take a theory T_0 having a non-principal type p . Take the lattice \mathcal{L} . For elements $a <_{\mathcal{L}} b$, say that if b “makes” p realised, so does a . For elements $c = \text{lub}_{\mathcal{L}}\{a, b\}$, say that if both a and b “make” p realised, then so does c . Satisfy these conditions in the freest possible way. Use amalgamation construction of course.

Since \mathcal{L}' is downward closed, there is an element a such that $\mathcal{L}' = \{x \mid x \leq_{\mathcal{L}} a\}$. Make the type corresponding to a decidable and leave undecidable as many types as possible. \square

Theorem (G, Khoussainov 2012)

Let \mathcal{L} be a finite lattice and \mathcal{L}' be its sublattice. Suppose \mathcal{L}' is downward closed in \mathcal{L} . Then there exists an AL theory T such that:

- 1 The fundamental order of T is \mathcal{L} , that is, $(\mathcal{AP}_T / \sim, \preceq) \cong \mathcal{L}$
- 2 The spectra of decibel models of T is \mathcal{L}' , that is, $\mathcal{AP}_T^D \cong \mathcal{L}'$.

Idea of proof.

Take a theory T_0 having a non-principal type p . Take the lattice \mathcal{L} . For elements $a <_{\mathcal{L}} b$, say that if b “makes” p realised, so does a . For elements $c = \text{lub}_{\mathcal{L}}\{a, b\}$, say that if both a and b “make” p realised, then so does c . Satisfy these conditions in the freest possible way. Use amalgamation construction of course.

Since \mathcal{L}' is downward closed, there is an element a such that $\mathcal{L}' = \{x \mid x \leq_{\mathcal{L}} a\}$. Make the type corresponding to a decidable and leave undecidable as many types as possible. \square

Theorem (G, Khoussainov 2012)

Let \mathcal{L} be a finite lattice and \mathcal{L}' be its sublattice. Suppose \mathcal{L}' is downward closed in \mathcal{L} . Then there exists an AL theory T such that:

- 1 The fundamental order of T is \mathcal{L} , that is, $(\mathcal{AP}_T / \sim, \preceq) \cong \mathcal{L}$
- 2 The spectra of decibel models of T is \mathcal{L}' , that is, $\mathcal{AP}_T^D \cong \mathcal{L}'$.

Idea of proof.

Take a theory T_0 having a non-principal type p . Take the lattice \mathcal{L} . For elements $a <_{\mathcal{L}} b$, say that if b “makes” p realised, so does a . For elements $c = \text{lub}_{\mathcal{L}}\{a, b\}$, say that if both a and b “make” p realised, then so does c . Satisfy these conditions in the freest possible way. Use amalgamation construction of course.

Since \mathcal{L}' is downward closed, there is an element a such that $\mathcal{L}' = \{x \mid x \leq_{\mathcal{L}} a\}$. Make the type corresponding to a decidable and leave undecidable as many types as possible. \square

Part 3

Spectra of computable models of AL theories

Question (Spectral problem in the class of computable presentations)

Let \mathcal{K} be a class of theories with fixed spectral parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of computable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of computable models?

- 1 \aleph_1 -categorical theories. The problem is quite complicated. An upper bound for the complexity of spectra is $\Sigma_3^0(\emptyset^\omega)$ (Nies). All known spectra are finite or co-finite (Many people from the US and Russia).
- 2 Ehrenfeucht theories. There are very many examples of the spectra. For instance, the spectra are not necessarily downward closed:

Theorem (G 2010)

There exists an Ehrenfeucht theory T and structures $A, B \in \mathcal{AP}_T$ such that $B \preceq A$, A is computable, B is not computable. Moreover, these structures can be chosen to be \sim -equivalent.

Question (Spectral problem in the class of computable presentations)

Let \mathcal{K} be a class of theories with fixed spectral parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of computable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of computable models?

- 1 \aleph_1 -categorical theories. The problem is quite complicated. An upper bound for the complexity of spectra is $\Sigma_3^0(\emptyset^\omega)$ (Nies). All known spectra are finite or co-finite (Many people from the US and Russia).
- 2 Ehrenfeucht theories. There are very many examples of the spectra. For instance, the spectra are not necessarily downward closed:

Theorem (G 2010)

There exists an Ehrenfeucht theory T and structures $A, B \in \mathcal{AP}_T$ such that $B \preceq A$, A is computable, B is not computable. Moreover, these structures can be chosen to be \sim -equivalent.

Question (Spectral problem in the class of computable presentations)

Let \mathcal{K} be a class of theories with fixed spectral parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of computable models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of computable models?

- 1 \aleph_1 -categorical theories. The problem is quite complicated. An upper bound for the complexity of spectra is $\Sigma_3^0(\emptyset^\omega)$ (Nies). All known spectra are finite or co-finite (Many people from the US and Russia).
- 2 Ehrenfeucht theories. There are very many examples of the spectra. For instance, the spectra are not necessarily downward closed:

Theorem (G 2010)

There exists an Ehrenfeucht theory T and structures $A, B \in \mathcal{AP}_T$ such that $B \preceq A$, A is computable, B is not computable. Moreover, these structures can be chosen to be \sim -equivalent.

The spectra are not necessarily intervals:

Theorem (G 2010)

There exists an Ehrenfeucht theory T and non \sim -equivalent structures $A, B, C \in \mathcal{AP}_T$ such that $A \preceq B \preceq C$, A and C are computable, B is not computable.

The situation in the class of computable presentations is completely different from the one in the class of decidable presentation:

Theorem (Khoussainov, Nies, Shore 1997)

There exists an Ehrenfeucht theory T having exactly 3 countable models such that the only computable model is saturated.

Spectra are not necessarily finite or cofinite:

Theorem (G 2011)

There exists an AL theory T that has infinite coinfinite suborder of \mathcal{AP}_T corresponding to computable models.

The spectra are not necessarily intervals:

Theorem (G 2010)

There exists an Ehrenfeucht theory T and non \sim -equivalent structures $A, B, C \in \mathcal{AP}_T$ such that $A \preceq B \preceq C$, A and C are computable, B is not computable.

The situation in the class of computable presentations is completely different from the one in the class of decidable presentation:

Theorem (Khoussainov, Nies, Shore 1997)

There exists an Ehrenfeucht theory T having exactly 3 countable models such that the only computable model is saturated.

Spectra are not necessarily finite or cofinite:

Theorem (G 2011)

There exists an AL theory T that has infinite coinfinite suborder of \mathcal{AP}_T corresponding to computable models.

Part 4

Spectra of automatic models of AL theories

Definition

A structure A of a finite predicate language is *automatic* if its domain and relations are recognisable by (finite string) automata.

Theorem (Hodgson 1976)

Automatic models are decidable.

Question (Spectral problem in the class of automatic presentations)

Let \mathcal{K} be a class of theories with fixed spectral parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of automatic models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of automatic models?

Theorem (Semukhin, Stephan 2010)

- 1 *There is an \aleph_1 -categorical theory the only automatic model of which is prime.*
- 2 *There is a small theory with automatic saturated model and non-automatic prime model.*

Definition

A structure A of a finite predicate language is *automatic* if its domain and relations are recognisable by (finite string) automata.

Theorem (Hodgson 1976)

Automatic models are decidable.

Question (Spectral problem in the class of automatic presentations)

Let \mathcal{K} be a class of theories with fixed spectral parameters $(\mathcal{AP}_T, \lambda_T)$. Describe the spectra of automatic models of these theories. In other words, what sub-parameters of the $(\mathcal{AP}_T, \lambda_T)$ can be realised as a spectrum of automatic models?

Theorem (Semukhin, Stephan 2010)

- 1 There is an \aleph_1 -categorical theory the only automatic model of which is prime.
- 2 There is a small theory with automatic saturated model and non-automatic prime model.

Theorem (G, Gavruskina 2011)

If an Ehrenfeucht theory T has infinite $dcl(\emptyset)$ and exactly 3 models, one of which is automatic, then all the models of T are automatic.

Proof.

Follows from the following two theorems. □

Theorem (Gavruskina 2010)

Let T be a variant of Ehrenfeucht's or Peretyatkin's example having an automatic model. Then all the models of T are automatic.

Theorem (Tanović 2007)

If an Ehrenfeucht theory T has infinite $dcl(\emptyset)$ and exactly 3 models, then T interprets a variant of Ehrenfeucht's or Peretyatkin's example.

Theorem (G, Gavruskina 2011)

If an Ehrenfeucht theory T has infinite $dcl(\emptyset)$ and exactly 3 models, one of which is automatic, then all the models of T are automatic.

Proof.

Follows from the following two theorems. □

Theorem (Gavruskina 2010)

Let T be a variant of Ehrenfeucht's or Peretyatkin's example having an automatic model. Then all the models of T are automatic.

Theorem (Tanović 2007)

If an Ehrenfeucht theory T has infinite $dcl(\emptyset)$ and exactly 3 models, then T interprets a variant of Ehrenfeucht's or Peretyatkin's example.

e-mail: a.gavruskin@auckland.ac.nz

Thank you for your attention!

these slides:

<http://sites.google.com/site/gavruskin/talks/2012NUS.pdf>